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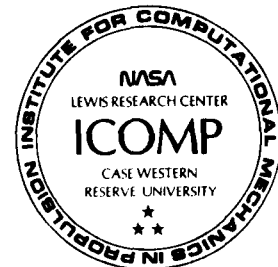
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1 Introduction

Realistic problems in acoustics and aerodynamics are typically time dependent and are often posed on unbounded spatial domains. Their numerical solution, of course, requires a bounded domain which may be obtained through the introduction of an artificial boundary. To complete the specification of the approximate problem, additional boundary conditions must be imposed there. For hyperbolic problems in particular, the behavior of the solution over long times can be strongly influenced by these additional conditions.

In the past decade many authors have developed rational approaches to the choice of boundary conditions at artificial boundaries. Most have been based on the asymptotic analysis of propagating waves. (For a general discussion of asymptotic expansions of waves see Ludwig [13].) There are (at least) two natural parameters which can be used to construct the expansion. One is the frequency, leading to the theory of geometrical optics. The construction of boundary conditions based on high frequency asymptotics was first described by Engquist and Majda [4] and has since been taken up by a number of others. Among the advantages of this approach are the following: geometrical optics is inherently local, leading always to local boundary conditions; inhomogeneities in the coefficients are easily handled; the construction of the expansions is well understood. A disadvantage, however, is that the expansion is subject to smooth errors which *may not be small*. That is, it can be difficult to estimate and control the accuracy of these procedures. Engquist and Halpern [3] have attempted to rectify this problem by combining the high frequency approximation with one valid for low frequencies. It is nonetheless unclear that the correction enhances the accuracy when the solution is far from steady-state.

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A second parameter which may be employed in the asymptotic analysis is the range of propagation, leading to a far-field expansion. The great advantage of boundary conditions based on far-field asymptotics is that their accuracy may be directly controlled either by taking more terms in the expansion or by extending the computational domain. A disadvantage is that far-field expansions are generally more difficult to compute and require more restrictions on the coefficients of the equation. For the wave equation, Friedlander [5] has developed an asymptotic expansion which in n dimensions is given by:

$$\phi \sim \sum_{j=0}^{\infty} \frac{f_j(t-r, \theta_1, \dots, \theta_{n-1})}{r^{j+\frac{n-1}{2}}}. \quad (1.0.1)$$

Here, r is a radial coordinate and the θ_i 's are angular coordinates. In [1], Bayliss and Turkel use this expansion to construct accurate boundary conditions.

In this work we construct expansions analagous to (1.0.1) for more general equations. In particular we consider problems which are governed in the far field by:

$$\frac{\partial^2 \phi}{\partial t^2} + \sum_{i=1}^n b_i \frac{\partial^2 \phi}{\partial x_i \partial t} - \sum_{i,j=1}^n a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0. \quad (1.0.2)$$

We assume constant coefficients and that the equation is hyperbolic. We further assume that the t -axis is a time-like direction and that all directions in the \vec{x} plane are space-like. These conditions are algebraically equivalent to the following: the $(n+1) \times (n+1)$ symmetric matrix C given by:

$$c_{ij} = \begin{cases} 1 & i=j=1 \\ \frac{b_i}{2} & j=1, i>1 \\ \frac{b_j}{2} & i=1, j>1 \\ -a_{ij} & i,j>1 \end{cases}, \quad (1.0.3)$$

has one positive and n negative eigenvalues and the $n \times n$ matrix A is positive definite. Deviations from this form such as inhomogeneities, nonlinearities, forcing terms and scattering bodies may be included in the analysis if they are restricted to a compact region in space, as must be the support of the initial data.

The outline of the remainder of the paper is as follows: in section 2 we derive a far-field expansion for (1.0.2). In section 3 we use it to derive asymptotic boundary conditions and analyze their stability and accuracy. In the final section we present two applications of the theory. The first application is to the study of unsteady vortical disturbances to the flow over a flat plate and the second is to the solution of the Euler equations for time-dependent flow past a cylinder. More detailed versions of aspects of this work will appear in [8], [9] and [10].

2 Asymptotic Expansions

We now consider the asymptotic analysis of solutions of (1.0.2) at *large distances*. We find it most convenient to work with the Riemann function, $S(\vec{x}, t)$, which is defined by:

$$LS = 0, \quad t > 0, \quad \vec{x} \in R^n, \quad (2.0.4)$$

$$S = 0, \quad S_t = \delta(\vec{x}), \quad t = 0. \quad (2.0.5)$$

Here, $\delta(\vec{x})$ is the Dirac delta function and we assume L is given by:

$$L = \left(\frac{\partial}{\partial t} + M \frac{\partial}{\partial x_1} \right)^2 - \nabla^2, \quad (2.0.6)$$

with $M < 1$. This is the convective wave operator (with subsonic convection). The general differential operator appearing in (1.0.2) may be put in this form by a change of spatial coordinates only. The restriction to coordinate changes which are independent of time is important for avoiding nonuniformities in the resulting expansions. To compute S , we temporarily introduce another change of coordinates:

$$\bar{t} = t, \quad (2.0.7)$$

$$y_1 = x_1 - Mt, \quad (2.0.8)$$

$$y_i = x_i, \quad i \neq 1. \quad (2.0.9)$$

With this change, L becomes the usual wave operator and the initial conditions for S become:

$$S = 0, \quad S_{\bar{t}} - MS_{y_1} = \delta(\vec{y}). \quad (2.0.10)$$

The first condition, however, implies that S_{y_1} is zero, so that the second reduces to $S_{\bar{t}} = \delta$. Therefore, S is the well-known Riemann function for the wave equation. In two spatial dimensions its expression in cylindrical coordinates is:

$$S(r, \theta, t) = \begin{cases} (1 - M^2)^{-\frac{1}{2}} (t - s(\theta)r)^{-\frac{1}{2}} (t + p(\theta)r)^{-\frac{1}{2}}, & t > s(\theta)r \\ 0, & t \leq s(\theta)r \end{cases}. \quad (2.0.11)$$

Here, p and s are given by:

$$s(\theta) = \frac{((1 - M^2 \sin^2 \theta)^{\frac{1}{2}} - M \cos \theta)}{(1 - M^2)}, \quad (2.0.12)$$

$$p(\theta) = \frac{((1 - M^2 \sin^2 \theta)^{\frac{1}{2}} + M \cos \theta)}{(1 - M^2)}. \quad (2.0.13)$$

An asymptotic expansion of S , valid for $\frac{(t-s(\theta)r)}{r}$ small, takes the form:

$$S \sim \sum_{j=0}^{\infty} a_j(\theta) \frac{(t - s(\theta)r)^{j-\frac{1}{2}}}{r^{j+\frac{1}{2}}}. \quad (2.0.14)$$

General solutions of (1.0.2) may be expressed in integral form (reference [13]):

$$\phi(\vec{x}, t) = \int_0^Z \int_0^{2\pi} \int_0^{t-s(\gamma(\vec{x}-\vec{y}))R(\vec{x}-\vec{y})} \mu(z, \psi, \eta) S(\vec{x} - \vec{y}, t - \eta) 2\pi z d\eta d\psi dz. \quad (2.0.15)$$

Here the domain of integration must include the support of the initial data, the inhomogeneous forcing terms, and other perturbations. The integration variables z and ψ are polar coordinates in \vec{y} and R and γ are the polar coordinates of $\vec{x} - \vec{y}$. We introduce:

$$r = t - s(\theta)r, \quad (2.0.16)$$

and note that:

$$t - sR = \tau - \Delta(\theta, \psi) - \epsilon(\vec{x}, \vec{y}), \quad (2.0.17)$$

where ϵ may be expanded:

$$\epsilon = \sum_{j=1}^{\infty} b_j(\theta, \psi) \left(\frac{z}{r} \right)^j. \quad (2.0.18)$$

By a change of integration variables and introduction of (2.0.14) the expression for ϕ becomes:

$$\int_0^Z \int_0^{2\pi} \int_0^{\tau - \Delta - \epsilon} \frac{\mu(z, \psi, \tau - \Delta - \epsilon - w)}{w^{\frac{1}{2}}(s(\theta) + p(\theta))^{\frac{1}{2}} r^{\frac{1}{2}}} \left(1 + \sum_{i+j \geq 1} c_{ij}(\theta, \psi) \frac{w^i z^j}{r^{i+j}} \right) 2\pi z dz d\psi dw. \quad (2.0.19)$$

Expanding in ϵ and using (2.0.18) we finally obtain the Friedlander form:

$$\phi \sim \sum_{j=0}^{\infty} \frac{f_j(t - s(\theta)r, \theta)}{r^{j+\frac{1}{2}}}. \quad (2.0.20)$$

A more rigorous exposition of the asymptotic analysis may be found in [8]. Some of the convergence analysis of [13] may also be applicable. We note that the functions f_j , $j \geq 1$ are determined by f_0 which is called the radiation field. Similar results hold for problems in three spatial dimensions. The Riemann function, expressed in spherical coordinates in \vec{x} space, is:

$$S(\vec{x}, t) = \frac{1}{2\pi(r^2 + M^2 t^2 - 2rMt \cos \phi \sin \theta)^{\frac{1}{2}}} H(t - (r^2 + M^2 t^2 - 2rMt \cos \phi \sin \theta)^{\frac{1}{2}}), \quad (2.0.21)$$

where H is the Heaviside function. This leads to an expansion for a general solution, u , of the form:

$$u(\vec{x}, t) \sim \sum_{j=0}^{\infty} \frac{f_j(t - s(\phi, \theta)r, \phi, \theta)}{r^{j+1}}, \quad (2.0.22)$$

$$s(\phi, \theta) = \frac{1}{1 - M^2} ((1 - M^2(1 - \sin^2 \theta \cos^2 \phi))^{\frac{1}{2}} - M \sin \theta \cos \phi). \quad (2.0.23)$$

3 Asymptotic Boundary Conditions

Given the expansions (2.0.20), (2.0.22) the derivation and analysis of asymptotic boundary conditions may essentially be copied from Bayliss and Turkel [1]. We introduce the m 'th order boundary operator, B_m :

$$B_m = \prod_{i=1}^m \left(\frac{\partial}{\partial r} + s(\theta) \frac{\partial}{\partial t} + \frac{(2i-1)}{2r} \right), \quad (3.0.24)$$

$$B_m \phi = O(r^{-\frac{4m+1}{2}}), \quad (3.0.25)$$

in two dimensions and

$$B_m = \prod_{i=1}^m \left(\frac{\partial}{\partial r} + s(\theta, \phi) \frac{\partial}{\partial t} + \frac{(2i-1)}{r} \right), \quad (3.0.26)$$

$$B_m \phi = O(r^{-(2m+1)}), \quad (3.0.27)$$

in three dimensions.

Standard arguments ([1]) lead to well-posedness and error estimates for a finite domain problem with the boundary condition:

$$B_m v = 0. \quad (3.0.28)$$

As we have $s(\theta) > 0$ ($s(\theta, \phi) > 0$), well-posedness follows directly from a theorem of Higdon [11], who in a later work [12] describes techniques for the construction of stable discretizations. As in [1], the equation may be used to replace the normal derivatives by tangential derivatives. In the numerical experiments which follow only B_1 is used.

If we let $e(\vec{x}, t)$ be the difference between the solution on the unbounded domain and the finite domain solution we have:

$$Le = 0, \quad t > 0, \quad \vec{x} \in \Omega \subset R^n, \quad (3.0.29)$$

$$e = e_t = 0, \quad t = 0, \quad (3.0.30)$$

$$B_m e = O(R^{-(2m + \frac{n-1}{2})}), \quad \vec{x} \in \partial\Omega, \quad (3.0.31)$$

where R is a measure of the size of Ω . The well-posedness implies that for an appropriate Sobolev norm:

$$\|e\|_{m,\Omega} \leq K_{m,\Omega} \|B_m e\|_{0,\partial\Omega}. \quad (3.0.32)$$

The scaling argument of [1] may be applied to our problem. It yields:

$$K_{m,\Omega} = O(R^{m + \frac{1}{2}}). \quad (3.0.33)$$

This finally yields the error estimate:

$$\|e\|_{m,\Omega} = O(R^{-(m + \frac{n}{2} - 1)}). \quad (3.0.34)$$

We note that the curves $r = cs$, c a constant, are the wavefronts defined by geometrical optics. Accordingly, boundary conditions based on high frequency asymptotics are related to those we have constructed. The first order boundary condition given by geometrical optics is:

$$(s \frac{\partial}{\partial t} + \frac{\partial}{\partial r}) \phi = 0, \quad (3.0.35)$$

for which we have:

$$\|Be\|_{0,\partial\Omega} = O(R^{-(1 + \frac{n-1}{2})}). \quad (3.0.36)$$

Further corrections, however, will in general differ from those we have computed.

4 Applications

4.1 Unsteady Vortical Disturbances Around a Flat Plate

The first application we consider is to the linearized analysis of unsteady vortical disturbances to the flow over a flat plate. The details of this study will appear in Hariharan, Ping and Scott [10]. This problem describes the influence of a transverse gust that impinges on an airfoil. The airfoil is taken to be a flat plate and the goal is to calculate the unsteady response function. Scott and Atassi [15] numerically compute the solution in the frequency domain, where the equations reduce to the Helmholtz equation. To calculate the response function, they performed separate calculations for each frequency. In contrast [10] deals directly with the time domain where the governing equation is a convective wave equation. Therefore the boundary condition proposed here may be directly

applied. A typical situation is depicted in Figure 1. The problem is symmetric about the line of the plate for uniform flows along the direction of the plate. The governing equations are:

$$\phi_{tt} + 2U_{\infty}\phi_{xt} + U_{\infty}^2\phi_{xx} = c_{\infty}^2\nabla^2\phi. \quad (4.1.37)$$

The boundary conditions along the wake line, on the plate and on the open boundaries respectively are

$$\left(\frac{\partial}{\partial t} + U_{\infty}\frac{\partial}{\partial x}\right)\phi = 0, \quad (4.1.38)$$

$$v + \frac{\partial\phi}{\partial y} = 0, \quad (4.1.39)$$

$$\phi_t + \frac{1}{s(\theta)}[\cos(\theta)\phi_x + \sin(\theta)\phi_y] + \frac{\phi}{2rs(\theta)} = 0. \quad (4.1.40)$$

Here v is the vertical component of the vortical disturbance and ϕ is the velocity potential. In addition the response function is defined by:

$$R(\omega) = \frac{G(\omega)}{F(\omega)}, \quad (4.1.41)$$

where $F(\omega)$ is the Fourier transform of the vortical input, $G(\omega)$ is the Fourier transform of the unsteady scaled lift given by

$$g(t) = \frac{L(t)}{\pi\rho c U_{\infty}} \quad (4.1.42)$$

and L is the lift over the plate calculated according to the formula

$$L = \int_{\Gamma} p ds. \quad (4.1.43)$$

Here Γ is the boundary of the plate. In this problem the vertical component of the transverse gust has the form

$$v(x, y, t) = f(x - t). \quad (4.1.44)$$

To compute results that correspond to various frequencies one can choose a Gaussian pulse with a wide band width. The results given in Figures 2a,b respectively correspond to solutions obtained by solving the above problem numerically. The results are compared to a benchmark cited in [15] for Mach numbers .5 and .8 respectively. That there is good agreement is clear. Furthermore, the time domain computations are far less costly than those in the frequency domain. For full details we refer readers to [10].

4.2 Euler Equations

As a second application we consider the Euler equations of inviscid gas dynamics. Using standard notation they are:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0, \quad (4.2.45)$$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p, \quad (4.2.46)$$

$$\rho \frac{De}{Dt} + p\nabla \cdot \vec{u} = 0. \quad (4.2.47)$$

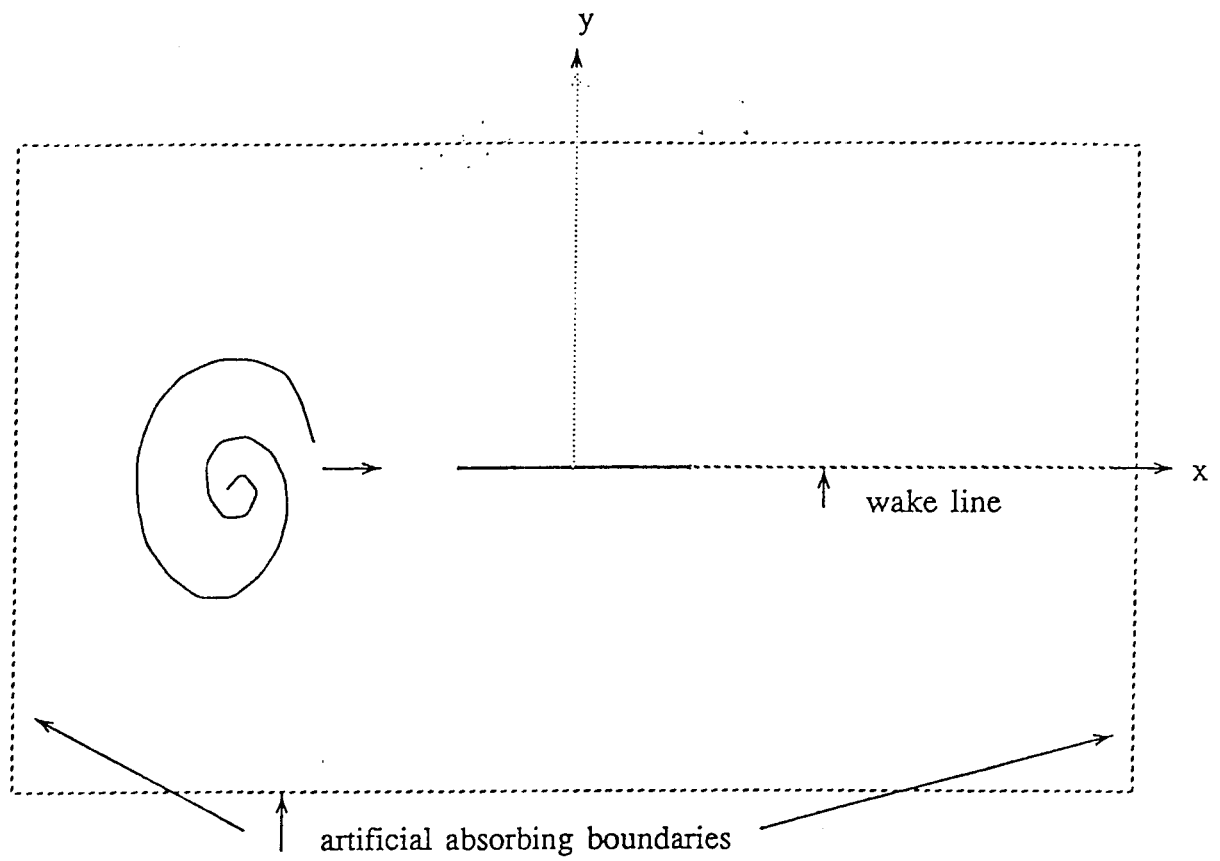


Figure 1: Typical flow configuration truncated by artificial boundaries.

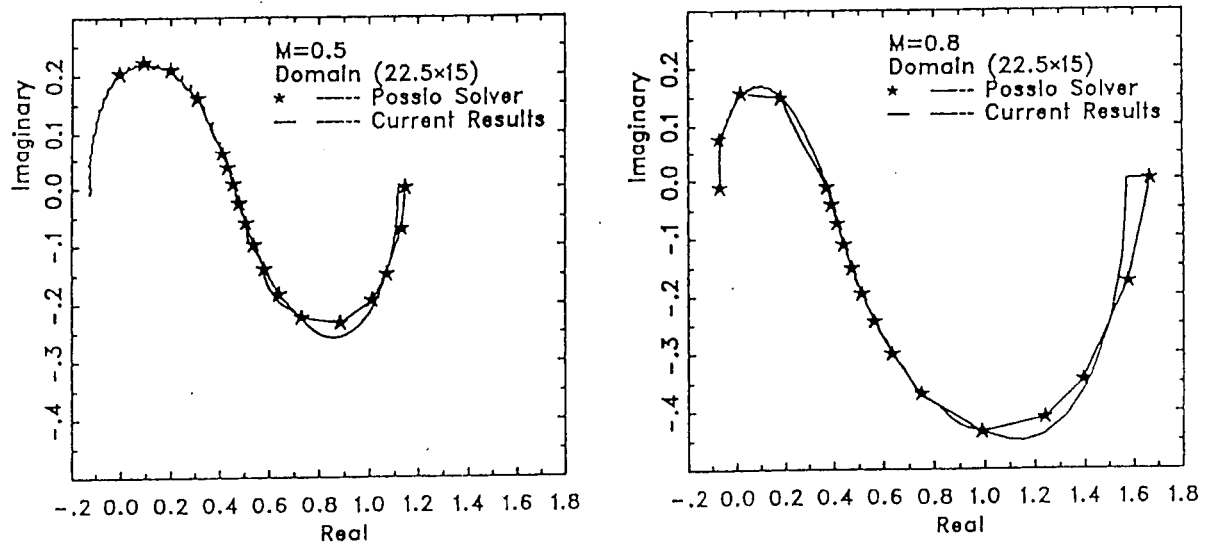


Figure 2: Comparison between the computed response function of a flat plate airfoil in a transverse gust at $M = .5$ and $M = .8$ and a Possio Solver (Ref. 15).

This system, which must be supplemented by an equation of state relating the density, ρ , the pressure, p , and the internal energy, e , is known to be hyperbolic.

We assume that in the far field the solution approaches a uniform subsonic flow in the x_1 direction and consider the linearization of (4.2.45-4.2.47) about the uniform state. We find that convenient variables for the analysis are the entropy, S , the vorticity, ω , the divergence of the velocity field, q , and the pressure, p . The equations then are:

$$\frac{\partial S}{\partial t} + u_\infty \frac{\partial S}{\partial x} = 0, \quad (4.2.48)$$

$$\frac{\partial \omega}{\partial t} + u_\infty \frac{\partial \omega}{\partial x} = 0, \quad (4.2.49)$$

$$\frac{\partial q}{\partial t} + u_\infty \frac{\partial q}{\partial x} + \frac{1}{\rho_\infty} \nabla^2 p = 0, \quad (4.2.50)$$

$$\frac{\partial p}{\partial t} + u_\infty \frac{\partial p}{\partial x} + \rho_\infty a_\infty^2 q = 0. \quad (4.2.51)$$

Clearly the equations for S and ω decouple. Their form implies that these variables should be imposed at inflow. A third boundary relation may be obtained from (4.2.50, 4.2.51). These observations concur with the conclusions of recent papers by Gustafsson [6] and Roe [14]. Combining equations (4.2.50) and (4.2.51) yields a convective wave equation for the pressure, to which we apply the asymptotic conditions discussed above.

In summary, we impose,

$$(s(\theta) \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{2r})(p - p_\infty) = 0, \quad (4.2.52)$$

at each point of the artificial boundary. These are supplemented at inflow by:

$$S = S_\infty, \quad (4.2.53)$$

$$\omega = 0. \quad (4.2.54)$$

We note that this pressure boundary condition has been previously proposed by Bayliss and Turkel [2] for use at a downstream boundary.

We briefly describe the results of a numerical experiment which demonstrates the accuracy of our approach. Details will appear in [9]. The (nonlinear) Euler equations were solved in the region exterior to a cylinder of radius one with initial conditions modeling an impulsive start. We employed a TVD scheme with the superbee flux limiter as part of a two-step Lax-Wendroff procedure (see Yee [16]). In Figure 3 the instantaneous density contours are shown for four solutions; two computed using the asymptotic boundary conditions and artificial boundary locations of $R = 5$ and 7 , one computed with the pressure specified and an artificial boundary at $R = 9$ and, for reference, one computed on a large domain, $R = 15$. The superiority of the results with the asymptotic condition is clear. We suspect that further improvements would be obtained with a nonlinear asymptotic analysis as was carried out in the isotropic case in [7].

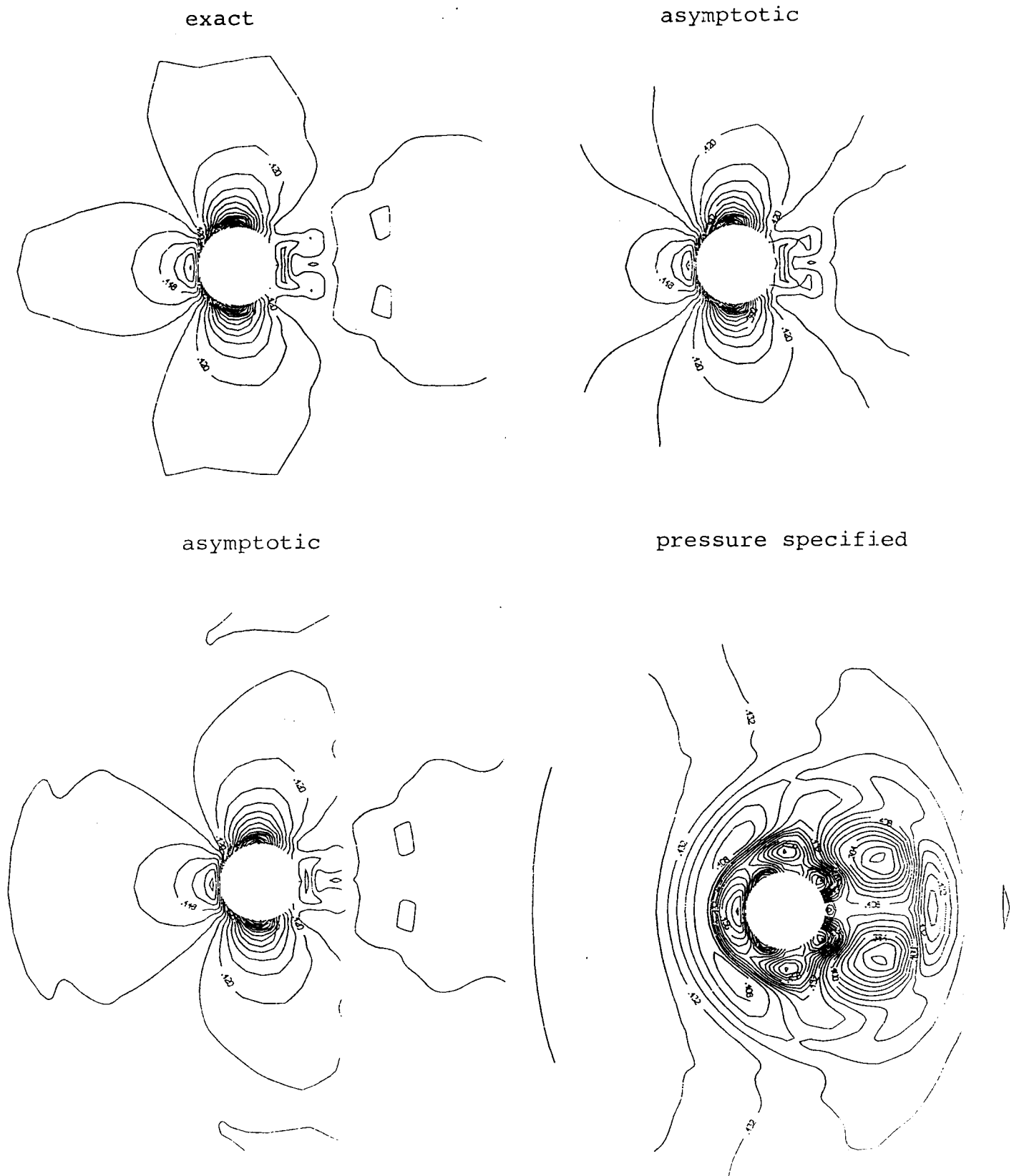


Figure 3: Density contours at $t = 20$ for an impulsively started cylinder with $M_\infty = .4$. The 'exact' condition is for a domain of radius 15, 'asymptotic' for a radii of 5 and 7, 'pressure specified' for a radius of 9.

References

- [1] A. Bayliss and E. Turkel, "Radiation boundary conditions for wave-like equations", *Comm. Pure Appl. Math.*, 33, (1980), 707-725.
- [2] A. Bayliss and E. Turkel, "Far field boundary conditions for compressible flows", *J. Comp. Phys.*, 48, (1982), 182-199.
- [3] B. Engquist and L. Halpern, "Far field boundary conditions for computation over long time", *Appl. Num. Math.*, 4, (1988), 21-45.
- [4] B. Engquist and A. Majda, "Absorbing boundary conditions for the numerical simulation of waves", *Math. Comp.*, 31, (1977), 629-651.
- [5] F.G. Friedlander, "On the radiation field of pulse solutions of the wave equation", *Proc. Roy. Soc. A*, 279, (1964), 386-394.
- [6] B. Gustafsson, "Far field boundary conditions for time-dependent hyperbolic systems", *SIAM J. Sci. Stat. Comp.*, 9, (1988), 812-828.
- [7] T. Hagstrom and S.I. Hariharan, "Accurate boundary conditions for exterior problems in gas dynamics", *Math. Comp.*, 51, (1988), 581-597.
- [8] T. Hagstrom and S.I. Hariharan, "Asymptotic expansions and accurate numerical radiation conditions for hyperbolic problems", in preparation.
- [9] T. Hagstrom and S.I. Hariharan, "Numerical simulation of time-dependent compressible flow in an exterior domain", in preparation.
- [10] S.I. Hariharan, Y. Ping and J.R. Scott, "Time Domain Numerical Calculations of Unsteady Vortical Flows About a Flat Plate Airfoil", submitted for publication.
- [11] R. Higdon, "Absorbing boundary conditions for difference approximations to the multi-dimensional wave equation", *Math. Comp.*, 47, (1986), 437-459.
- [12] R. Higdon, "Numerical absorbing boundary conditions for the wave equation", *Math. Comp.*, 49, (1987), 65-90.
- [13] D. Ludwig, "Exact and asymptotic solutions of the Cauchy problem", *Comm. Pure Appl. Math.*, 13, (1960), 473-508.
- [14] P. Roe, "Remote boundary conditions for unsteady multidimensional aerodynamic computations", ICASE Report No. 86-75, (1986).
- [15] J.R. Scott and H.M. Atassi, "Numerical solution of periodic vortical flows about a thin airfoil", AIAA Paper 89-1691, June 1989.
- [16] H.C. Yee, "Construction of explicit and implicit symmetric TVD schemes and their applications", *J. Comp. Phys.*, 68, (1987), 151-179.



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16. Abstract A necessary ingredient for the numerical simulation of many time dependent phenomena in acoustics and aerodynamics is the imposition of accurate radiation conditions at artificial boundaries. The asymptotic analysis of propagating waves provides a rational approach to the development of such conditions. In this work we derive a far field asymptotic expansion of solutions of anisotropic wave equations. This generalizes the well known Friedlander expansion for the standard wave operator. We use the expansion to derive a hierarchy of radiation conditions of increasing accuracy. Two numerical experiments are given to illustrate the utility of our approach. The first application is to the study of unsteady vortical disturbances impinging on a flat plate; the second is to the simulation of inviscid flow past an impulsively started cylinder.					
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